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ON A FUNDAMENTAL SOLUTION OF ONE LINEAR ORDINARY DIFFERENTIAL EQUATION WITH FRACTIONAL ORDER DERIVATIVE

Allahshukur Guliyev*

Lankaran State University, Lankaran, Azerbaijan

Abstract. In the paper we consider contraction of the fundamental solution of one linear ordinary differential equation with fractional derivative. The order of the derivative is positive and less than two. By the factorization method we obtain fundamental solution for the equation from the first order linear ordinary differential equation with constant coefficients.

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Corresponding author: Allahshukur Guliyev, Lankaran State University, Lankaran, Azerbaijan, e-mail:
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1 Introduction

It is known that in many papers the Cauchy and boundary value problem is considered for the ordinary differential equation (Nagy, 2021; Irvii, 2021). For the partial differential equation the Cauchy and mixed problems have been considered for hyperbolic and parabolic type equations, and boundary value problems are mainly considered for the elliptic type equations (Chow, 2003; Richtmayer, 1982).

In the case of ordinary differential equations the number of boundary conditions is taken equal to the order of the given equation (Mammadov & Khankishiyev, 2013), for the partial differential equations the number of initial conditions is taken equal to the order of the derivative with respect to the time variable included in the given equation, and the number of boundary conditions in the arbitrary smooth boundary domain is taken to be equal to the half of the highest order derivative with respect to the spatial variable (Garling, 2013).

For the Laplace equation one Dirichlet, Neumann, or Poincaré condition condition is given (Irvii, 2021). For a biharmonic equation (the fourth order equation), two boundary conditions are given (Rectoris, 1985). These conditions are called local conditions. In (Aliyev et al., 2012) is shown the Fredholm property of the problem with non-local boundary conditions obtained by dividing the boundary into two parts for the first-order elliptic type Cauchy-Riemann equation and then combining the boundary values of the sought function.

This eliminates the misunderstanding in the boundary value problems for ordinary and partial differential equations (Aliyev et al., 2012). Thus, if the boundary is divided into two parts, then the number of non-local boundary conditions is equal to the order of the given equation, if the boundary is divided into four parts, then the number of non-local boundary conditions is twice the order of the equation (Aliyev et al., 2010). But then other difficulties arise. Thus, if more than one point moves on the border at the same time, then the Carleman condition must be satisfied. That is, the moving points on the boundary must either move away from or approach a point on the boundary. Otherwise, the obtained problem is not correct

(Aliyev et al., 2012).

In (Aliyev et al., 2012) is shown that if the boundary for the Cauchy-Riemann equation is divided into two parts, and the Carleman condition is satisfied, then the boundary value problem is reduced to the second type of Fredholm integral equation with the necessary conditions, and if the Carleman condition is not satisfied, to the first type Fredholm integral equation.

In the presented work the fundamental solution is obtained for the linear equation of order less that two with constant coefficients from the solution of the linear second order differential equation with constant coefficients by using the factorization method.

Note that the solution of the considered boundary value problem is related to the contraction the Green function. The construction of the Green function is a difficult problem itself. But the construction of the fundamental solution, which is its main part, is not related to the boundary condition, but only depends on the equation. So the equation can be considered in entire space. In this regard, the book (Vladimirov, 1981) should be noted, where the construction of the fundamental solutions is described in detail.

2 Problem formulation

Consider the problem

$$D^2y(x) + aDy(x) + by(x) = f(x), \tag{1}$$

where $D = \frac{d}{dx}$, $a,\ b$ are given numbers, $f(x)$ is a given continues function.

It is easy to construct the general solution of the corresponding homogeneous equation

$$y''(x) + ay'(x) + by(x) = 0 \tag{2}$$

in the form

$$y(x) = \sum_{k=1}^{2} C_k y_k(x), \tag{3}$$

where C_k are arbitrary constants.

To find these constants one can use the method of variation of constants and get

$$C_1(x) = C_1 - \int_0^x \frac{e^{\varphi_2\,\xi} f(\xi)}{\mathrm{W}(\xi)} d\xi, \qquad C_2(x) = C_2 + \int_0^x \frac{e^{\varphi_1\,\xi} f(\xi)}{\mathrm{W}(\xi)} d\xi.$$

So, we get

$$y(x) = \sum_{k=1}^{2} C_k e^{\varphi_k\,x} + \int_0^x \frac{e^{\varphi_2\,x+\varphi_1\,\xi} - e^{\varphi_1\,x+\varphi_2\,\xi}}{\mathrm{W}(\xi)} f(\xi)d\xi =$$

$$= \sum_{k=1}^{2} C_k e^{\varphi_k\,x} + \frac{1}{\sqrt{a^2-4b}} \int_0^x \left[e^{\varphi_2\,x+(\varphi_1+a)\,\xi} - e^{\varphi_1\,x+(\varphi_2+a)\,\xi} \right] f(\xi)d\xi =$$

$$= \sum_{k=1}^{2} C_k e^{\varphi_k\,x} + \int_0^x \frac{e^{\varphi_2\,(x-\xi)} - e^{\varphi_1\,(x-\xi)}}{\sqrt{a^2-4b}} f(\xi)d\xi. \tag{4}$$

Thus we get the following formula for the fundamental solution of equation (1)

$$Y(x) = \frac{e^{\varphi_2\,x} - e^{\varphi_1\,x}}{\sqrt{a^2-4b}} \theta(x). \tag{5}$$

Here $\theta(x)$ is Heviside's unit function.

Now we factorize the operator corresponding to equation (1) and present it as a product of two equations with fractional order derivative. Then we will construct the fundamental solution for the linear, ordinary differential equation of order $\frac{3}{2}$ with constant coefficients.

$$D^2 + aD + b = (D^{\frac{3}{2}} + \alpha D + \beta D^{\frac{1}{2}} + \gamma)(D^{\frac{1}{2}} + \sigma) =$$

$$= D^2 + \sigma D^{\frac{3}{2}} + \alpha D^{\frac{3}{2}} + \alpha\sigma D + \beta D + \beta\sigma D^{\frac{1}{2}} + \gamma D^{\frac{1}{2}} + \gamma\sigma. \tag{6}$$

For the factorization coefficients we have the system below

$$\begin{cases} \sigma + \alpha = 0, & \sigma = -\alpha, \\ \alpha\sigma + \beta = a, & \beta = a + \alpha^2, \\ \beta\sigma + \gamma = 0, & \gamma = a\alpha + \alpha^3, \\ \gamma\sigma = b, & -a\alpha^2 - \alpha^4 = b. \end{cases}$$

The last gives

$$\alpha = \pm\sqrt{\frac{-a \pm \sqrt{a^2 - 4b}}{2}}. \tag{7}$$

We choose one of the four values of αobtained from (7)

$$\alpha = \sqrt{\frac{-a + \sqrt{a^2 - 4b}}{2}}. \tag{8}$$

Then

$$\alpha = \sqrt{\frac{-a + \sqrt{a^2 - 4b}}{2}}, \quad \sigma = -\sqrt{\frac{-a + \sqrt{a^2 - 4b}}{2}}, \quad \beta = \frac{a + \sqrt{a^2 - 4b}}{2}, \quad \gamma = \alpha \cdot \frac{a + \sqrt{a^2 - 4b}}{2}.$$

From here we derive

$$\alpha = \sqrt{\varphi_2}, \qquad \sigma = -\sqrt{\varphi_2}, \qquad \beta = -\varphi_1, \qquad \gamma = -\varphi_1\sqrt{\varphi_2} = -\frac{b}{\sqrt{\varphi_2}}. \tag{9}$$

Thus

$$(D^2 + aD + b)Y(x) = (D^{\frac{3}{2}} + \sqrt{\varphi_2}D - \varphi_1 D^{\frac{1}{2}} - \varphi_1\sqrt{\varphi_2})(D^{\frac{1}{2}} - \sqrt{\varphi_2})Y(x) = \delta(x). \tag{10}$$

Set the denotation

$$Z(x) = (D^{\frac{1}{2}} - \sqrt{\varphi_2})Y(x). \tag{11}$$

Then from (10) we get

$$D^{\frac{3}{2}}Z(x) + \sqrt{\varphi_2}DZ(x) - \varphi_1 D^{\frac{1}{2}}Z(x) - \varphi_1\sqrt{\varphi_2}Z(x) = \delta(x). \tag{12}$$

Now we calculate $Z(x)$considering (12)

$$Z(x) = D^{\frac{1}{2}}Y(x) - \sqrt{\varphi_2}Y(x) = \frac{1}{\sqrt{a^2 - 4b}}\left[D^{\frac{1}{2}}\left(e^{\varphi_2 x}\theta(x)\right) - D^{\frac{1}{2}}\left(e^{\varphi_1 x}\theta(x)\right)\right] -$$

$$-\frac{\sqrt{\varphi_2}}{\sqrt{a^2 - 4b}}\left[e^{\varphi_2 x}\theta(x) - e^{\varphi_1 x}\theta(x)\right] =$$

$$= \frac{1}{\sqrt{a^2 - 4b}}\left\{I_{-\infty}^{\frac{1}{2}}\left(\varphi_2 e^{\varphi_2 x}\theta(x) + e^{\varphi_2 x}\delta(x) - \varphi_1 e^{\varphi_1 x}\theta(x) - e^{\varphi_1 x}\delta(x)\right)\right\} -$$

$$-\frac{\sqrt{\varphi_2}}{\sqrt{a^2-4b}}\left[e^{\varphi_2 x}\theta(x)-e^{\varphi_1 x}\theta(x)\right]=\frac{\varphi_2}{\sqrt{a^2-4b}}\int_{-\infty}^{x}\frac{(x-t)^{-\frac{1}{2}}}{(-\frac{1}{2})!}e^{\varphi_2 t}\theta(t)dt-$$

$$-\frac{\varphi_1}{\sqrt{a^2-4b}}\int_{-\infty}^{x}\frac{(x-t)^{-\frac{1}{2}}}{(-\frac{1}{2})!}e^{\varphi_1 t}\theta(t)dt-\frac{\sqrt{\varphi_2}}{\sqrt{a^2-4b}}\left[e^{\varphi_2 x}\theta(x)-e^{\varphi_1 x}\theta(x)\right].$$

Thus

$$Z(x)=\frac{\varphi_2}{\sqrt{a^2-4b}}\int_{0}^{x}\frac{(x-t)^{-\frac{1}{2}}}{(-\frac{1}{2})!}e^{\varphi_2 t}dt-\frac{\varphi_1}{\sqrt{a^2-4b}}\int_{0}^{x}\frac{(x-t)^{-\frac{1}{2}}}{(-\frac{1}{2})!}e^{\varphi_1 t}dt-$$
$$-\frac{\sqrt{\varphi_2}}{\sqrt{a^2-4b}}\left[e^{\varphi_2 x}\theta(x)-e^{\varphi_1 x}\theta(x)\right], \tag{13}$$

where $x>0$ is assumed.

Now we show that $Z(x)$ satisfies (12). To do this we can write

$$D^{\frac{3}{2}}Z(x)+\sqrt{\varphi_2}DZ(x)-\varphi_1 D^{\frac{1}{2}}Z(x)-\varphi_1\sqrt{\varphi_2}Z(x)=$$

$$=D\left\{\frac{\varphi_2}{\sqrt{a^2-4b}}\,D\int_0^x\frac{(x-\tau)^{-\frac{1}{2}}}{(-\frac{1}{2})!}d\tau\int_0^\tau\frac{(\tau-t)^{-\frac{1}{2}}}{(-\frac{1}{2})!}e^{\varphi_2 t}dt-\right.$$

$$-\frac{\varphi_1}{\sqrt{a^2-4b}}D\int_0^x\frac{(x-\tau)^{-\frac{1}{2}}}{(-\frac{1}{2})!}d\tau\int_0^\tau\frac{(\tau-t)^{-\frac{1}{2}}}{(-\frac{1}{2})!}e^{\varphi_1 t}dt-$$

$$-\frac{\sqrt{\varphi_2}}{\sqrt{a^2-4b}}\left[D\int_0^x\frac{(x-\tau)^{-\frac{1}{2}}}{(-\frac{1}{2})!}e^{\varphi_2\tau}\theta(\tau)d\tau-\right.$$

$$\left.\left.-D\int_0^x\frac{(x-\tau)^{-\frac{1}{2}}}{(-\frac{1}{2})!}e^{\varphi_1\tau}\theta(\tau)d\tau\right]\right\}+\frac{\varphi_2\sqrt{\varphi_2}}{\sqrt{a^2-4b}}D\int_0^x\frac{(x-t)^{-\frac{1}{2}}}{(-\frac{1}{2})!}e^{\varphi_2 t}dt-$$

$$-\frac{\varphi_1\sqrt{\varphi_2}}{\sqrt{a^2-4b}}D\int_0^x\frac{(x-t)^{-\frac{1}{2}}}{(-\frac{1}{2})!}e^{\varphi_1 t}dt-\frac{\varphi_2}{\sqrt{a^2-4b}}\left[\varphi_2e^{\varphi_2 x}\theta(x)+e^{\varphi_2 x}\delta(x)-\varphi_1e^{\varphi_1 x}\theta(x)-e^{\varphi_1 x}\delta(x)\right]-$$

$$-\frac{\varphi_1\varphi_2}{\sqrt{a^2-4b}}D\int_0^x\frac{(x-\tau)^{-\frac{1}{2}}}{(-\frac{1}{2})!}d\tau\int_0^\tau\frac{(\tau-t)^{-\frac{1}{2}}}{(-\frac{1}{2})!}e^{\varphi_2\tau}d\tau+$$

$$+\frac{\varphi_1^2}{\sqrt{a^2-4b}}D\int_0^x\frac{(x-\tau)^{-\frac{1}{2}}}{(-\frac{1}{2})!}d\tau\int_0^\tau\frac{(\tau-t)^{-\frac{1}{2}}}{(-\frac{1}{2})!}e^{\varphi_1\tau}d\tau+$$

$$+\frac{\varphi_1\sqrt{\varphi_2}}{\sqrt{a^2-4b}}\left[D\int_0^x\frac{(x-\tau)^{-\frac{1}{2}}}{(-\frac{1}{2})!}e^{\varphi_2\tau}\theta(\tau)d\tau-D\int_0^x\frac{(x-\tau)^{-\frac{1}{2}}}{(-\frac{1}{2})!}e^{\varphi_1\tau}\theta(\tau)d\tau\right]-$$

$$-\frac{\varphi_1\varphi_2\sqrt{\varphi_2}}{\sqrt{a^2-4b}}\int_0^x\frac{(x-t)^{-\frac{1}{2}}}{(-\frac{1}{2})!}e^{\varphi_2 t}dt+\frac{\varphi_1^2\sqrt{\varphi_2}}{\sqrt{a^2-4b}}\int_0^x\frac{(x-t)^{-\frac{1}{2}}}{(-\frac{1}{2})!}e^{\varphi_1 t}dt+$$
$$\frac{\varphi_1\varphi_2}{\sqrt{a^2-4b}}\left[e^{\varphi_2 x}\theta(x)-e^{\varphi_1 x}\theta(x)\right]. \tag{14}$$

We change the integration turn here

$$\int_0^x\frac{(x-\tau)^{-\frac{1}{2}}}{(-\frac{1}{2})!}d\tau\int_0^\tau\frac{(\tau-t)^{-\frac{1}{2}}}{(-\frac{1}{2})!}e^{\phi_k t}dt=\int_0^x e^{\phi_k t}dt\int_t^x\frac{(x-\tau)^{-\frac{1}{2}}}{(-\frac{1}{2})!}\frac{(\tau-t)^{-\frac{1}{2}}}{(-\frac{1}{2})!}d\tau \tag{15}$$

and consider the replacement

$$x-\tau=\eta(x-t) \tag{16}$$

in the internal. Since $\tau = x - \eta(x-t)$ we obtain $\tau - t = (x-t)(1-\eta)$. Then (15) gives

$$\int_0^x \frac{(x-\tau)^{-\frac{1}{2}}}{(-\frac{1}{2})!} d\tau \int_0^\tau \frac{(\tau-t)^{-\frac{1}{2}}}{(-\frac{1}{2})!} e^{\varphi_k t} dt =$$

$$= -\int_0^x e^{\varphi_k t} dt \int_1^0 \frac{\eta^{-\frac{1}{2}}(x-t)^{-\frac{1}{2}}}{(-\frac{1}{2})!} \frac{(x-t)^{-\frac{1}{2}}(1-\eta)^{-\frac{1}{2}}}{(-\frac{1}{2})!}(x-t) d\eta =$$

$$= \int_0^x e^{\varphi_k t} dt \int_0^1 \frac{\eta^{-\frac{1}{2}}(1-\eta)^{-\frac{1}{2}}}{\left[(-\frac{1}{2})!\right]^2} d\eta =$$

$$= \int_0^x e^{\varphi_k t} dt = \left.\frac{e^{\varphi_k t}}{\varphi_k}\right|_{t=0}^x = \tfrac{e^{\varphi_k x} - 1}{\varphi_k}, \quad k = 1, 2. \tag{17}$$

Subsituting (17) into (14) we obtain

$$\left(D^{\frac{3}{2}} + \sqrt{\varphi_2} D - \varphi_1 D^{\frac{1}{2}} - \varphi_1 \sqrt{\varphi_2}\right) Z(x) =$$

$$= D\left\{ \frac{\varphi_2}{\varphi_2 - \varphi_1} e^{\varphi_2 x} - \frac{\varphi_1}{\varphi_2 - \varphi_1} e^{\varphi_1 x} - \frac{\sqrt{\varphi_2}}{\varphi_2 - \varphi_1} \left[D \int_0^x \frac{(x-\tau)^{-\frac{1}{2}}}{(-\frac{1}{2})!} e^{\varphi_2 \tau} d\tau - \right.\right.$$

$$\left.\left. - D \int_0^x \frac{(x-\tau)^{-\frac{1}{2}}}{(-\frac{1}{2})!} e^{\varphi_1 \tau} d\tau \right] \right\} + \frac{(\varphi_2 + \varphi_1)\sqrt{\varphi_2}}{\varphi_2 - \varphi_1} D \int_0^x \frac{(x-\tau)^{-\frac{1}{2}}}{(-\frac{1}{2})!} e^{\varphi_2 \tau} d\tau -$$

$$- \frac{2\varphi_1 \sqrt{\varphi_2}}{\varphi_2 - \varphi_1} D \int_0^x \frac{(x-\tau)^{-\frac{1}{2}}}{(-\frac{1}{2})!} e^{\varphi_1 \tau} d\tau -$$

$$- \frac{\varphi_2^2}{\varphi_2 - \varphi_1} e^{\varphi_2 x} \theta(x) + \frac{\varphi_1 \varphi_2}{\varphi_2 - \varphi_1} e^{\varphi_1 x} \theta(x) - \frac{\varphi_1 \varphi_2}{\varphi_2 - \varphi_1} e^{\varphi_2 x} +$$

$$+ \frac{\varphi_1^2}{\varphi_2 - \varphi_1} e^{\varphi_1 x} - \frac{\varphi_1 \varphi_2 \sqrt{\varphi_2}}{\varphi_2 - \varphi_1} \int_0^x \frac{(x-t)^{-\frac{1}{2}}}{(-\frac{1}{2})!} e^{\varphi_2 t} dt +$$

$$+ \frac{\varphi_1^2 \sqrt{\varphi_2}}{\varphi_2 - \varphi_1} \int_0^x \frac{(x-t)^{-\frac{1}{2}}}{(-\frac{1}{2})!} e^{\varphi_1 t} dt + \frac{\varphi_1 \varphi_2}{\varphi_2 - \varphi_1} e^{\varphi_2 x} \theta(x) - \frac{\varphi_1 \varphi_2}{\varphi_2 - \varphi_1} e^{\varphi_1 x} \theta(x). \tag{18}$$

We transform the integrals included in (18) as follows

$$\int_0^x \frac{(x-\tau)^{-\frac{1}{2}}}{(-\frac{1}{2})!} e^{\varphi_k \tau} d\tau = -\int_0^x e^{\varphi_k \tau} d_\tau \left(\frac{(x-\tau)^{\frac{1}{2}}}{\frac{1}{2}!} \right) =$$

$$= -\left[\left. e^{\varphi_k \tau} \frac{(x-\tau)^{\frac{1}{2}}}{\frac{1}{2}!} \right|_{\tau=0}^x - \varphi_k \int_0^x e^{\varphi_k \tau} \frac{(x-\tau)^{\frac{1}{2}}}{\frac{1}{2}!} d\tau \right] =$$

$$= \frac{x^{\frac{1}{2}}}{\frac{1}{2}!} + \varphi_k \int_0^x e^{\varphi_k \tau} \frac{(x-\tau)^{\frac{1}{2}}}{\frac{1}{2}!} d\tau = \frac{x^{\frac{1}{2}}}{\frac{1}{2}!} - \varphi_k \int_0^x e^{\varphi_k \tau} d_\tau \left(\frac{(x-\tau)^{\frac{3}{2}}}{\frac{3}{2}!} \right) =$$

$$= \frac{x^{\frac{1}{2}}}{\frac{1}{2}!} - \varphi_k \left[\left. e^{\varphi_k \tau} \frac{(x-\tau)^{\frac{3}{2}}}{\frac{3}{2}!} \right|_{\tau=0}^x - \varphi_k \int_0^x e^{\varphi_k \tau} \frac{(x-\tau)^{\frac{3}{2}}}{\frac{3}{2}!} d\tau \right] = \tag{19}$$

$$= \frac{x^{\frac{1}{2}}}{\frac{1}{2}!} + \varphi_k \frac{x^{\frac{3}{2}}}{\frac{3}{2}!} + \varphi_k^2 \int_0^x e^{\varphi_k \tau} \frac{(x-\tau)^{\frac{3}{2}}}{\frac{3}{2}!} d\tau, \quad k = 1, 2.$$

Putting (19) into the doubt integrals in (18) we get

$$-\frac{\sqrt{\varphi_2}}{\varphi_2-\varphi_1}D^2\left[\frac{x^{\frac{1}{2}}}{\frac{1}{2}!}+\varphi_2\frac{x^{\frac{3}{2}}}{\frac{3}{2}!}+\varphi_2^2\int_0^x e^{\varphi_2\tau}\frac{(x-\tau)^{\frac{3}{2}}}{\frac{3}{2}!}d\tau-\frac{x^{\frac{1}{2}}}{\frac{1}{2}!}-\varphi_1\frac{x^{\frac{3}{2}}}{\frac{3}{2}!}-\varphi_1^2\int_0^x e^{\varphi_1\tau}\frac{(x-\tau)^{\frac{3}{2}}}{\frac{3}{2}!}d\tau\right]=$$

$$=-\sqrt{\varphi_2}\frac{x^{-\frac{1}{2}}}{(-\frac{1}{2})!}-\frac{\varphi_2^2\sqrt{\varphi_2}}{\varphi_2-\varphi_1}\int_0^x e^{\varphi_2\tau}\frac{(x-\tau)^{-\frac{1}{2}}}{(-\frac{1}{2})!}d\tau+\frac{\varphi_1^2\sqrt{\varphi_2}}{\varphi_2-\varphi_1}\int_0^x e^{\varphi_1\tau}\frac{(x-\tau)^{-\frac{1}{2}}}{(-\frac{1}{2})!}d\tau. \qquad (20)$$

By the same way putting (19) into the single integrals in (18) gives

$$\frac{(\varphi_2+\varphi_1)\sqrt{\varphi_2}}{\varphi_2-\varphi_1}\left[\frac{x^{-\frac{1}{2}}}{(-\frac{1}{2})!}+\varphi_2\int_0^x\frac{(x-\tau)^{-\frac{1}{2}}}{(-\frac{1}{2})!}e^{\varphi_2\tau}d\tau\right]-$$

$$-\frac{2\varphi_1\sqrt{\varphi_2}}{\varphi_2-\varphi_1}\left[\frac{x^{-\frac{1}{2}}}{(-\frac{1}{2})!}+\varphi_1\int_0^x\frac{(x-\tau)^{-\frac{1}{2}}}{(-\frac{1}{2})!}e^{\varphi_1\tau}d\tau\right]-$$

$$-\frac{\varphi_1\varphi_2\sqrt{\varphi_2}}{\varphi_2-\varphi_1}\int_0^x\frac{(x-\tau)^{-\frac{1}{2}}}{(-\frac{1}{2})!}e^{\varphi_2\tau}d\tau+\frac{\varphi_1^2\sqrt{\varphi_2}}{\varphi_2-\varphi_1}\int_0^x\frac{(x-\tau)^{-\frac{1}{2}}}{(-\frac{1}{2})!}e^{\varphi_1\tau}d\tau=$$

$$=\sqrt{\varphi_2}\frac{x^{-\frac{1}{2}}}{(-\frac{1}{2})!}+\frac{\varphi_2^2\sqrt{\varphi_2}}{\varphi_2-\varphi_1}\int_0^x\frac{(x-\tau)^{-\frac{1}{2}}}{(-\frac{1}{2})!}e^{\varphi_2\tau}d\tau+\frac{\varphi_1\varphi_2\sqrt{\varphi_2}}{\varphi_2-\varphi_1}\int_0^x\frac{(x-\tau)^{-\frac{1}{2}}}{(-\frac{1}{2})!}e^{\varphi_2\tau}d\tau-$$

$$-\frac{2\varphi_1^2\sqrt{\varphi_2}}{\varphi_2-\varphi_1}\int_0^x\frac{(x-\tau)^{-\frac{1}{2}}}{(-\frac{1}{2})!}e^{\varphi_1\tau}d\tau-\frac{\varphi_1\varphi_2\sqrt{\varphi_2}}{\varphi_2-\varphi_1}\int_0^x\frac{(x-\tau)^{-\frac{1}{2}}}{(-\frac{1}{2})!}e^{\varphi_2\tau}d\tau+$$

$$+\frac{\varphi_1^2\sqrt{\varphi_2}}{\varphi_2-\varphi_1}\int_0^x\frac{(x-\tau)^{-\frac{1}{2}}}{(-\frac{1}{2})!}e^{\varphi_1\tau}d\tau=$$

$$=\sqrt{\varphi_2}\frac{x^{-\frac{1}{2}}}{(-\frac{1}{2})!}+\frac{\varphi_2^2\sqrt{\varphi_2}}{\varphi_2-\varphi_1}\int_0^x\frac{(x-\tau)^{-\frac{1}{2}}}{(-\frac{1}{2})!}e^{\varphi_2\tau}d\tau-\frac{\varphi_1^2\sqrt{\varphi_2}}{\varphi_2-\varphi_1}\int_0^x\frac{(x-\tau)^{-\frac{1}{2}}}{(-\frac{1}{2})!}e^{\varphi_1\tau}d\tau.$$

Putting the obtained relations into (18) we finally get

$$\left(D^{\frac{3}{2}}+\sqrt{\varphi_2}D-\varphi_1D^{\frac{1}{2}}-\varphi_1\sqrt{\varphi_2}\right)Z(x)=-\sqrt{\varphi_2}\frac{x^{-\frac{1}{2}}}{(-\frac{1}{2})!}-\frac{\varphi_2^2\sqrt{\varphi_2}}{\varphi_2-\varphi_1}\int_0^x e^{\varphi_2\tau}\frac{(x-\tau)^{-\frac{1}{2}}}{(-\frac{1}{2})!}d\tau+$$

$$+\frac{\varphi_1^2\sqrt{\varphi_2}}{\varphi_2-\varphi_1}\int_0^x e^{\varphi_1\tau}\frac{(x-\tau)^{-\frac{1}{2}}}{(-\frac{1}{2})!}d\tau+\sqrt{\varphi_2}\frac{x^{-\frac{1}{2}}}{(-\frac{1}{2})!}+\frac{\varphi_2^2\sqrt{\varphi_2}}{\varphi_2-\varphi_1}\int_0^x e^{\varphi_2\tau}\frac{(x-\tau)^{-\frac{1}{2}}}{(-\frac{1}{2})!}d\tau-$$

$$-\frac{\varphi_1^2\sqrt{\varphi_2}}{\varphi_2-\varphi_1}\int_0^x e^{\varphi_1\tau}\frac{(x-\tau)^{-\frac{1}{2}}}{(-\frac{1}{2})!}d\tau-\frac{\varphi_2^2}{\varphi_2-\varphi_1}e^{\varphi_2x}\theta(x)-\frac{\varphi_1\varphi_2}{\varphi_2-\varphi_1}e^{\varphi_2x}+\frac{\varphi_1^2}{\varphi_2-\varphi_1}e^{\varphi_1x}+$$

$$+\frac{\varphi_1\varphi_2}{\varphi_2-\varphi_1}e^{\varphi_2x}\theta(x)+\frac{\varphi_2^2}{\varphi_2-\varphi_1}e^{\varphi_2x}-\frac{\varphi_1^2}{\varphi_2-\varphi_1}e^{\varphi_1x}\equiv 0. \qquad (21)$$

Here $x>0$ is assumed. Relation (21) is valid also for $x<0$.

Thus, using the fundamental solution of the second order linear ordinary differential equation with constant coefficients we obtained fundamental solution for the $\frac{3}{2}$ order linear ordinary differential equation with constant coefficients.

Namely, the following theorem is proved.

Theorem 1. *The fundamental solution of* $\frac{3}{2}$ *order equation* (12) *obtained by factorization* (10) *of the second order linear equation* (1) *with constant coefficients is in the form of* (13).

The approach given above makes possible to obtain the fundamental solution for the linear ordinary differential equations with constant coefficients of fractional order less than two using different factorizations of equation (1).

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